Some algebraic structures in the KPZ Universality

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Overview

1. Examples of discrete models

2. Robinson-Schensted-Knuth correspondence

3. A geometric lifting of RSK - Kirillov's "Tropical RSK"

4. The totally asymmetric simple exclusion process

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Kardar-Parisi-Zhang equation

Definition 1.1 (Kardar-Parisi-Zhang equation)

The Kardar-Parisi-Zhang equation is defined by

$$\frac{\partial h}{\partial t} = \frac{1}{2}\triangle h + \frac{1}{2}|\nabla h|^2 + \dot{W}(t,x), \qquad x \in \mathbb{R}^d, t > 0. \tag{1}$$

$\dot{W}$ is the space-time white noise, which is a distribution valued Gaussian process, delta correlated in space and time as

$$E[\dot{W}(t,x)\dot{W}(t,x)] = \delta(t-s)\delta(x-y).$$

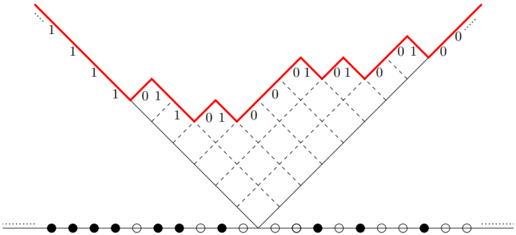
Asymptotically for large time t,the solution (1) behaves as

$$h(t,x) \approx \mu t + t^{1/3}\eta(\frac{x}{t^{2/3}}), \tag{2}$$

where μ is a macroscopic constant and $\eta(\cdot)$ is a random function.

Corner growth

Let us map the features of the **corner growth process** to the terms of (1).



- The fact that unit squares fill corners is consistent with the smoothing effect of the Laplacian.
- Parts of the interface which are very stretched, grow slower than other parts with many corners. This is consistent with the growth of the interface being proportional to $|\frac{\partial h}{\partial x}|^2$.
- The randomness and independence of the waiting times until corners are filled is consistent with the presence of $\dot{W}$.

Corner growth

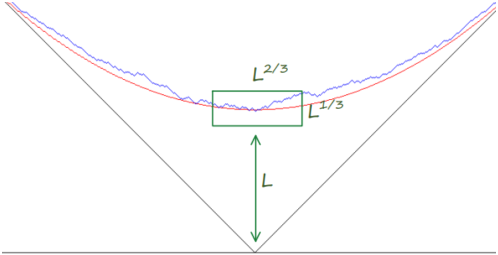
Each corner turns into an unit square after an exponential rate 1 waiting time.
Define h(t,x) = height above x at time t. Wedge initial data is h(0,x)=|x|.

Theorem 1.2 [Rost 1981]

For wedge initial data as t grows,

$$\frac{h(t,tx)}{t} \xrightarrow{t\to+\infty} \begin{cases} \dfrac{1-x^2}{2} & ,|x|<1 \\ |x| & ,|x|\geqslant 1 \end{cases}. \tag{3}$$

Define the rescaled height function $h_L(t,x) = L^{-1/3}[h(Lt, L^{2/3}x) - \frac{Lt}{2}]$.

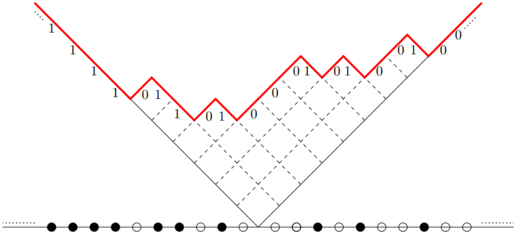


Theorem1.3 [Johansson 1999]

For wedge initial data as L grows,

$$\mathbf{P}(h_L(t,0) > -s) \to F_{GUE}(s). \qquad (4)$$

We can map the interface and the dynamics of the corner growth process to an interacting particle system



This configuration can also be projected onto the one dimensional lattice $\frac{1}{2}\mathbb{Z}$, with 1's corresponding to particles and 0's corresponding to empty sites.

The waiting times are exponentially distributed, then this particle process is Markovian and known as the **Totally Asymmetric Exclusion Process (TASEP)**.

For each $x \in \frac{1}{2}\mathbb{Z}$ let us denote by

$$\zeta_t(x) := \mathbf{1}_{\{a \text{ particle occupies site } x \text{ at time } t\}}.$$

Then we see that

$$\zeta_t(x) = \frac{h(t,x) - h(t,x-1) + 1}{2}.$$

Last Passage Percolation

We define two signs as follows:

- $\tau_{x,y}$: time for the corner growth interface to cover a corner with bottom site (x, y)
- $\omega_{x,y}$: time for bottom site (x, y) is covered once neighbouring sites (x−1, y−1) and (x+1, y−1) are covered.

Now we have the recursive equation

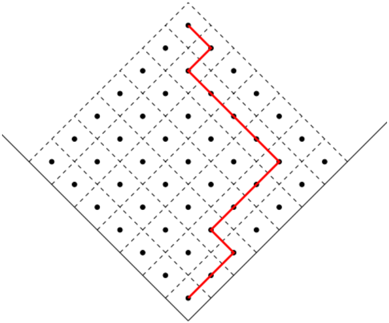
$$\tau_{x,y} = \max(\tau_{x-1,y-1}, \tau_{x+1,y-1}) + \omega_{x,y}. \tag{5}$$

Iterating this and denoting by $\prod_{x,y}$ the set of directed, up-left or up-right paths from (1,1) to (x,y), we derive the variational formula

$$\tau_{x,y} = \max_{\pi \in \prod_{x,y}} \sum_{v \in \pi} \omega_v. \tag{6}$$

Last Passage Percolation

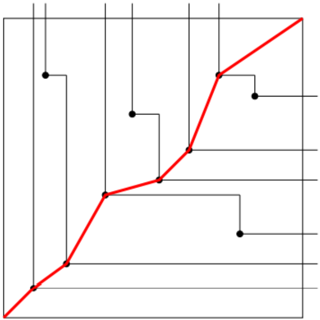
This is depicted in the following figure:



The quantity (6) is known as **Last Passage Percolation time** and its statistics are linked to the statistics of the height via

$$\mathbb{P}(h(t,x) \geqslant y) = \mathbb{P}(\tau_{x,y} \leqslant t).$$

Now we consider the degenerate last passage percolation problem: In the square with side length N, and lower-left corner (0, 0) we have a Poisson Point Process with intensity 1 and we ask what is the maximum number of Poisson points(the length of the longest path) that can be collected by going from (0, 0) to (N, N) via an up-right path.
This is known as **Hammersley problem**.



We can read the length of such maximal path as follows: From each point draw a horizontal and vertical line going rightwards and upwards, respectively. If two such rays meet, they cancel each other.

From the above figure, we can see that **the length of the longest path equals the number of rays that reach either the top or right side of the square**.

Hammersley Process and Longest Increasing Subsequenses

We can map the Hammersley problem to the problem of **longest increasing subsequence** in a random permuation as follows:

- Order the horizontal and vertical coordinates of the Poisson points in the square according to the order of their projections.
- Write the coordinates (x,y) of each point in the form of a **biletter**$\binom{x}{y}$.
- the length of the longest upright path through these points equals the length of the longest increasing subsequence in the permutation.

In the example of the above figure, we represent all the points in the form of a double array as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 10 & 2 & 4 & 7 & 5 & 6 & 9 & 3 & 8 \end{pmatrix}$$

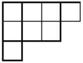
So the length of the longest upright path is 6.

Overview

Young tableau

- A **partition** of a number n is a sequence of non-increasing numbers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ such that $\lambda_1 + \lambda_2 + \cdots = n$.
- A partition can be depicted by **Young diagrams**. These are arrays of left justified unit boxes, the first row of which has λ_1 boxes, the second row λ_2 boxes etc.For example:

$$\lambda = (4,3,1) \qquad \longleftrightarrow$$



- The boxes in a Young diagram are usually filled with (integer) numbers giving rise to either a standard **Young tableau**,if the content of the boxes are strictly increasing along rows and columns, or a **semistandard Young tableau**, if the contents are strictly increasing along columns but weakly increasing along rows.
- The vector $(\lambda_1, \lambda_2, \cdots)$ of the lengths of the rows of the Young tableau T is called the **shape** of the tableau and we denote it by **sh(T)**.

Robinson-Schensted correspondence

Consider a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix},$$

where we denote $x_i := \sigma(i)$.

The **Robinson-Schensted(RS) algorithm** gives a one-to-one correspondence between a permutation $\sigma \in S_N$ and a pair of standard Young tableaux, which we will denote by (P, Q).It works as follows:

- Starting from a pair of empty tableaux $(P_0, Q_0) = (\emptyset, \emptyset)$, assume that we have inserted the first i **biletters** $\binom{i}{x_i}, 1 \leqslant i \leqslant N$,of the permutation $\sigma \in S_N$ and we have obtained a pair of Young tableaux(P_i, Q_i).

Robinson-Schensted correspondence

- Next, we **insert** the biletter $\binom{i+1}{x_{i+1}}$ as follows:
  If the number x_{i+1} is larger or equal than all the numbers of the first row of P_i, then a box is appended at the end of the first row of Pi and its content is set to be x_{i+1}. This is the tableau P_{i+1}. Also a box is appended at the end of the first row of Q_i and its content is set to be $i+1$, giving the tableau Q_{i+1}.
  If there is a box in the firstrow of P_i with content strictly larger than x_{i+1}, then the content of the first such box becomes x_{i+1} and the replaced content drops down and is row inserted in the second row of P_i following the same rules and creating (possibly) a cascade of dropdowns (called **bumps**). Eventually a box will be appended at the end of a row in Pi or below its last row, in which case it creates a new row, and the content of this box will be the last bumped letter. At the same, corresponding location a box will be added at Q_i and its content will be set to be $i+1$.
- We repeat the above steps until all biletters have been row inserted.

An example:

- permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 6 & 2 & 4 & 7 \end{pmatrix},$$

- sequence:

$$(\emptyset,\emptyset) \xrightarrow{3} \boxed{3}\;\; \boxed{1} \xrightarrow{5} \boxed{3\,|\,5}\;\; \boxed{1\,|\,2} \xrightarrow{1} \begin{array}{ll} 1 & 5 \\ 3 & \end{array}\;\; \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \xrightarrow{6} \begin{array}{lll} 1 & 5 & 6 \\ 3 & & \end{array}\;\; \begin{array}{lll} 1 & 2 & 4 \\ 3 & & \end{array} \xrightarrow{2}$$

$$\xrightarrow{2} \begin{array}{lll} 1 & 2 & 6 \\ 3 & 5 & \end{array}\;\; \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & \end{array} \xrightarrow{4} \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}\;\; \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array} \xrightarrow{7} \begin{array}{llll} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & \end{array}\;\; \begin{array}{llll} 1 & 2 & 4 & 7 \\ 3 & 5 & 6 & \end{array}.$$

Some observations:

- Tableaux P and Q have the same shape.
- Tableaux P and Q are actually equal.
- The length of the first row of tableaux P/Q equals the length of the longest increasing subsequence in the permutation.
  Moreover, the length of the second row of the output tableaux equals the length of the second longest increasing subsequence.

Robinson-Schensted correspondence

Actually, the second observation is not a general fact but a consequence of the fact that the permutation is symmetric, i.e $\sigma = \sigma^{-1}$.

The above observations are summarised in the following theorems:

Theorem 2.1 (Schensted)

The **RS** correspondence is a bijection between permutations and pairs of standard Young tableaux (P, Q) of the same shape. If $\sigma \in S_N$ and $(P, Q) = \mathbf{RS}(\sigma)$ is the image of σ under **RS**-correspondence, then $(Q, P) = \mathbf{RS}(\sigma^{-1})$, where σ^{-1} is the inverse of permutation σ. In particular, if $\sigma = \sigma^{-1}$, then $P = Q$.

Robinson-Schensted correspondence

Theorem 2.2 (Greene)

Let $\sigma \in S_N$ and $(P, Q) = \mathbf{RS}(\sigma)$.Then, the length λ_1 of the first row of the output tableaux P or Q equals the length of the longest increasing subsequence in σ. Moreover, the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ of the lengths of the first r rows equals the maximum possible length of unions of r increasing subsequences in σ.

Matrice $\longleftrightarrow$ Pairs of **seminstandard Young tableaux**

where matrices with nonnegative integer entries.

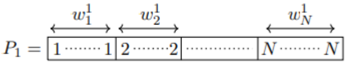
Let the matrix $W = (w_j^i)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant N}$, where i indicates rows and j columns, as a sequence of n words

$$w^i := 1^{w_1^i} 2^{w_2^i} \cdots N^{w_N^i} := \underbrace{1\cdots 1}_{w_1^i}\underbrace{2\cdots 2}_{w_2^i}\cdots\underbrace{N\cdots N}_{w_N^i} \tag{7}$$

Robinson-Schensted-Knuth correspondence

The Young tableau P is constructed as follows:

- Start by inserting w^1,the insertion of w^1 will produce the one-row tableau



  the tableau can be identified with the single word

$$p^1 := 1^{p_1^1} 2^{p_2^1} \cdots N^{p_N^1} = 1^{w_1^1} 2^{w_2^1} \cdots N^{w_N^1} = w^1.$$

- Next, we insert word w^2 into P_1 and this insertion will produce a new tableau $\tilde{P}_1$. We denote this schematically as

$$P_1 \overset{w^2}{\longrightarrow} \tilde{P}_1.$$

Robinson-Schensted-Knuth correspondence

- This insertion will change the first row p^1 of P_1 by (possibly) bumping some letters out of it and replacing them with letters from w^2. The bumped letters will form a word, which will then be inserted in the second row of the tableau. We denote this schematically as

$$p^1 \xrightarrow[\textstyle\downarrow v^2]{w^2} \tilde{p}^1,$$

with p^1 denoting the first row of P_1, $\tilde{p}^1$ the first row of $\tilde{P}_1$ and v^2 the word that will form from the bumped down letters from P_1 after the insertion of w^2.

Robinson-Schensted-Knuth correspondence

- The following picture is a building block of RSK, since the row insertion of a word w in a tableau P consisting of rows $p_1, p_2, \cdots, p_n$, can be decomposed as($w := v^1$)

$$p^1 \xrightarrow{v^1} \tilde{p}^1$$

$$p^2 \xrightarrow{v^2} \tilde{p}^2$$

$$\vdots$$

$$p^n \xrightarrow[v^{n+1}]{v^n} \tilde{p}^n$$

  It means that the letters that will drop down from p^1, after the insertion of $v^1 = w$, will form a word v^2 which will be inserted in p^2, forming a new row $\tilde{p}^2$ and so on.

- **Remark**:Row p^i only include letters with value larger or equal to i.

Robinson-Schensted-Knuth correspondence

Proposition 2.3 (**RSK** row insertion)

Let $1 \leqslant i \leqslant N$. Consider two words $\mathbf{x} = i^{x_i}(i+1)^{x_{i+1}} \cdots N^{x_N}$ and $\mathbf{a} = i^{a_i}(i+1)^{a_{i+1}} \cdots N^{a_N}$ The row insertion of the word $\mathbf{a}$ into the word $\mathbf{x}$ denoted by

$$\mathbf{x} \underset{\mathbf{b}}{\overset{\mathbf{a}}{\downarrow\!\!\longrightarrow}} \tilde{\mathbf{x}}$$

transforms $(\mathbf{x}, \mathbf{a})$ into a new pair $(\tilde{\mathbf{x}}, \mathbf{b})$ with $\tilde{\mathbf{x}} = i^{\tilde{x}_i}(i+1)^{\tilde{x}_{i+1}} \cdots N^{\tilde{x}_N}$ and $\mathbf{b} = (i+1)^{b_{i+1}} \cdots N^{b_N}$, which in cumulative variables (for $j \geqslant i$) $\xi_j = x_i + x_{i+1} + \cdots + x_j, \tilde{\xi}_j = \tilde{x}_i + \tilde{x}_{i+1} + \cdots + \tilde{x}_j$ is encoded via (for $i < N$)

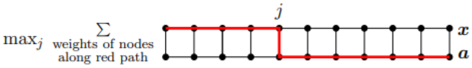
$$\begin{aligned}
\tilde{\xi}_i &= \xi_i + a_i \\
\tilde{\xi}_k &= \max(\tilde{\xi}_{k-1}, \xi_k) + a_k, && i+1 \leqslant k \leqslant N \\
b_k &= a_k + (\xi_k - \xi_{k-1}) - (\tilde{\xi}_k - \tilde{\xi}_{k-1}), && i+1 \leqslant k \leqslant N
\end{aligned} \tag{8}$$

Robinson-Schensted-Knuth correspondence

An observation: The recursion $\tilde{\xi}_k = \max(\tilde{\xi}_{k-1}, \xi_k) + a_k$ is actually the same as the recursion of last passage percolation (5). Now let i=1, We iterate as

$$\begin{aligned}
\tilde{\xi}_N &= \max(\tilde{\xi}_{N-1} + a_N, \xi_N + a_N) \\
&= \max(\max(\tilde{\xi}_{N-2} + a_{N-1} + a_N, \xi_{N-1} + a_{N-1} + a_N), \xi_N + a_N) \\
&\vdots \\
&= \max_{1 \leqslant j \leqslant N} (\xi_j + a_j + \cdots + a_N) \\
&= \max_{1 \leqslant j \leqslant N} (x_1 + \cdots + x_j + a_j + \cdots + a_N),
\end{aligned} \tag{9}$$

which, as shown in the figure below, is a last passage percolation on a two-row array



(10)

Gelfand-Tsetlin parametrisation

Gelfand-Tsetlin (GT) patterns are triangular arrays of number $(z_j^i)_{1 \leqslant j \leqslant i \leqslant N}$, which interlace, meaning that $z_{j+1}^{i+1} \leqslant z_j^i \leqslant z_j^{i+1}$,and for this reason they are depicted as

$$
\begin{array}{ccccccccc}
 & & & & z_1^1 & & & & \\
 & & & z_2^2 & & z_1^2 & & & \\
 & & z_3^3 & & z_2^3 & & z_1^3 & & \\
 & & & & \vdots & & & & \\
z_N^N & & z_{N-1}^n & & \cdots & & z_2^N & & z_1^N .
\end{array}
$$

They provide a particularly useful parametrisation of Young tableaux: given a Young tableau consisting of letters $1, 2, \cdots N$, the Gelfand-Tsetlin variables z_j^i are defined as

$$z_j^i := \sum_{k=j}^{i} \# \left\{ k\text{'s in the } j^{th} \text{ row} \right\}$$

Gelfand-Tsetlin parametrisation

In GT parametrisation,We define the vector

$$(|z^i| - |z^{i-1}| : i = 1, \ldots, N), \text{ with } |z^i| := \textstyle\sum_{j=1}^{i} z_j^i \ ,$$

and the convention that $|z^0| = 0$.Considering a pair of GT patterns $(\mathbf{Z}, \mathbf{Z'})$ as the output of **RSK** with input matrix $W = (w_j^i)_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant N}$, that is $(\mathbf{Z}, \mathbf{Z'}) = \mathbf{RSK}(W)$, with $\mathbf{Z}$ corresponding to the P tableau and $\mathbf{Z}$ to the Q tableau in the **RSK** correspondence. Now we have the fact as follows:

$$\begin{aligned}
|z^k| - |z^{k-1}| &= \sum_{j=1}^{k} z_j^k - \sum_{j=1}^{k-1} z_j^{k-1} = z_k^k - \sum_{j=1}^{k-1} (z_j^k - z_j^{k-1}) \\
&= \#\left\{k\text{'s in the } k^{th} \text{ row}\right\} + \sum_{j=1}^{k-1} \#\left\{k\text{'s in the } j^{th} \text{ row}\right\} \quad (11) \\
&= \sum_{i=1}^{n} w_k^i .
\end{aligned}$$

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- As we have seen in Proposition 2.3, **RSK** can be encoded in terms of piecewise linear recursive relations, using the (max, $+$) algebra. Kirillov replaced the (max, $+$) in the set of **RSK**'s piecewise linear relations with relations ($+$, $\times$), thus establishing a **geometric lifting of RSK(gRSK)**.
- In this section we will present the construction of gRSK following mostly a matrix reformulation by Noumi and Yamada motivated by discrete integrable systems. The approach is closely related to that of Proposition 3.3. Let us start with the definition of the geometric row insertion.

Geometric RSK via a matrix formulation

Definition 3.1

Let $1 \leqslant i \leqslant N$. Consider two words $\mathbf{x} = (x_i, x_{i+1}, \cdots, x_N)$ and $\mathbf{a} = (a_i, a_{i+1}, \cdots, a_N)$ We define the **geometric lifting of row insertion**(**geometric row insertion**) of the word $\mathbf{a}$ into the word $\mathbf{x}$ denoted by

$$\mathbf{x} \overset{\mathbf{a}}{\underset{\mathbf{b}}{\longrightarrow}} \tilde{\mathbf{x}}$$

transforms $(\mathbf{x}, \mathbf{a})$ into a new pair $(\tilde{\mathbf{x}}, \mathbf{b})$ with $\tilde{\mathbf{x}} = (\tilde{x}_i, \tilde{x}_{i+1}, \cdots, \tilde{x}_N)$ and $\mathbf{b} = (b_{i+1}, \cdots, b_N)$, which in cumulative variables (for $j \geqslant i$) $\xi_j = x_i x_{i+1} \cdots x_j$, $\tilde{\xi}_j = \tilde{x}_i \tilde{x}_{i+1} \cdots \tilde{x}_j$ is encoded via (for $i < N$)

$$\begin{aligned} \tilde{\xi}_i &= \xi_i a_i \\ \tilde{\xi}_k &= a_k(\tilde{\xi}_{k-1} + \xi_k), i+1 \leqslant k \leqslant N \\ b_k &= a_k \frac{\xi_k \tilde{\xi}_{k-1}}{\xi_{k-1}\tilde{\xi}_k}, \qquad i+1 \leqslant k \leqslant N \end{aligned} \tag{12}$$

Geometric RSK via a matrix formulation

From relations (12), we can get the following equations:

$$
\begin{aligned}
a_i x_i &= \tilde{x}_i \\
a_j x_j &= \tilde{x}_j b_j & \quad \textit{for } j \geqslant i+1 \\
\frac{1}{a_i} + \frac{1}{x_{i+1}} &= \frac{1}{b_{i+1}} \\
\frac{1}{a_j} + \frac{1}{x_{j+1}} &= \frac{1}{\tilde{x}_j} + \frac{1}{b_{j+1}} & \quad \textit{for } j \geqslant i+1
\end{aligned}
\tag{13}
$$

The derivation of the system of equations in (13) from (12) is a matter of a simple algebraic manipulation. (13) can be put into a matrix form as($\overline{x} := \frac{1}{x}$):

Geometric RSK via a matrix formulation

$$
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & \overline{a}_i & 1 & & & \\
& & & \overline{a}_{i+1} & \ddots & & \\
& & & & \ddots & 1 & \\
& & & & & \overline{a}_N &
\end{pmatrix}
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & \overline{x}_i & 1 & & & \\
& & & \overline{x}_{i+1} & \ddots & & \\
& & & & \ddots & 1 & \\
& & & & & \overline{x}_N &
\end{pmatrix}
=
$$

$$
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & \bar{\tilde{x}}_i & 1 & & & \\
& & & \bar{\tilde{x}}_{i+1} & \ddots & & \\
& & & & \ddots & 1 & \\
& & & & & \bar{\tilde{x}}_N &
\end{pmatrix}
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & 1 & 1 & & & \\
& & & \overline{b}_{i+1} & \ddots & & \\
& & & & \ddots & 1 & \\
& & & & & \overline{b}_N &
\end{pmatrix}
\quad (※)
$$

Geometric RSK via a matrix formulation

For the convenience of expression, we define the matrix

$$E_i(\mathbf{x}) := \sum_{j=1}^{i-1} E_{jj} + \sum_{j=i}^{N} x_j E_{jj} + \sum_{j=i}^{N-1} E_{j,j+1}, \text{where } \mathbf{x} = (x_i, x_{i+1}, \cdots, x_n).$$

Then we write the (※) as

$$E_i(\bar{\mathbf{a}})E_i(\bar{\mathbf{x}}) = E_i(\bar{\tilde{\mathbf{x}}})E_{i+1}(\bar{\mathbf{b}}) \quad (14)$$

$E_i(\mathbf{x})$ can be readily read graphically from the following diagram:

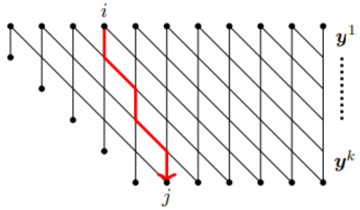
where on the diagonal edges and on the first (i−1) vertical edges we assign the value 1 and on the rest of the vertical edges we assign the values $x_i, x_{i+1}, \cdots, x_N$ in this order.

Geometric RSK via a matrix formulation

(Continued from above)
The (k, l) entry of $E_i(\mathbf{x})$ is given by$E_i(\mathbf{x})_{(k,l)} = \sum_{\pi:(1,k)\to(2,l)} \mathbf{wt}(\pi)$, where the sum is over all down-right paths, along existing edges, starting from site k in the top row to site l in the bottom row and the weight of the path π is given by the product of the weights along the edges that path π traces. Now we define the general form

$$E(\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^k) := E_1(\mathbf{y}^1)E_2(\mathbf{y}^2)\cdots E_k(\mathbf{y}^k), \text{where } \mathbf{y}^i = (y^i_i, y^i_{i+1}, \cdots, y^i_N).$$

The entries can be read graphically from the following diagram:



(Continued from above)
where a vertical edge connecting (a, b) to $(a+1, b)$ is assigned the weight y^a_b and all the diagonal edges are assigned weight one.

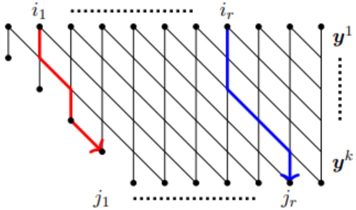
The (i, j) entry of $E(\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^k)$ is given by $E(\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^k)_{(i,j)}$ $=\sum_{\pi:(1,i)\to(k\wedge j+1,j)} \mathbf{wt}(\pi)$, where the sum is over all down-right paths, along existing edges, from site $(1, i)$ in the top row to site $(k \wedge j + 1, j)$ along the lower border and the weight of a path $\mathbf{wt}(\pi) = \prod_{\mathbf{e}\in\pi} w_{\mathbf{e}}$, where the product is over all edges $\mathbf{e}$ that are traced by the path π.

Geometric RSK via a matrix formulation

Let rows $i_1 < \cdots < i_r$ and columns $j_1 < \cdots < j_r$. We can get the following equation:

$$\det E(\mathbf{y}^1, \cdots, \mathbf{y}^k)^{i_1,\cdots,i_r}_{j_1,\cdots,j_r} = \sum_{\pi_1 \cdots \pi_r \in \prod^{i_1,\cdots,i_r}_{j_1,\cdots,j_r}} \mathbf{wt}(\pi_1) \cdots \mathbf{wt}(\pi_r) \qquad (15)$$

where $\prod^{i_1,\cdots,i_r}_{j_1,\cdots,j_r}$ is the set of directed, non intersecting paths, starting at locations $i_1, \cdots, i_r$ in the top row and ending at locations $j_1, \cdots, j_r$ at the bottom border of the grid:



Geometric RSK via a matrix formulation

The equation (15) is a consequence of the following Theorem:

Theorem 3.2 [Lindström-Gessel-Viennot]

Let $G = (V, E)$ be a directed, acyclic graph with no multiple edges, with each edge $\mathbf{e}$ being assigned a weigth $\mathbf{wt(e)}$. A path π on G is assigned a weight $\mathbf{wt}(\pi) = \prod_{\mathbf{e}\in\pi} w_{\mathbf{e}}$. We say that two paths on G are non-intersect ing if they do not share any vertex. Consider, now, $(u_1, \cdots, u_r)$ and $(v_1, \cdots, v_r)$ two disjoint subsets of V and denote by $\prod_{v_1,\cdots,v_r}^{u_1,\cdots,u_r}$ the set of all r-tuples of non-intersecting paths $\pi_1, \cdots, \pi_r$ that start from $u_1, \cdots, u_r$ and end at $v_1, \cdots, v_r$ respectively. We assume that $u_1, \cdots, u_r$ and $v_1, \cdots, v_r$ have the property that for $i < i\,'$ and $j > j\,'$, any two paths $\pi \in \prod_{u_i}^{v_j}$ and $\pi' \in \prod_{u_{i'}}^{v_{j'}}$, which start at $u_i, u_{i'}$ and end at $v_j, v_{j'}$, necessarily intersect. Then

$$\det\Big(\sum_{\pi\in\prod_{v_j}^{u_i}} \mathbf{wt}(\pi)\Big)_{1\leqslant i,j\leqslant r} = \sum_{\pi_1\cdots\pi_r\in\prod_{j_1,\cdots,j_r}^{i_1,\cdots,i_r}} \mathbf{wt}(\pi_1)\cdots\mathbf{wt}(\pi_r) \tag{16}$$

Geometric RSK via a matrix formulation

Let us start with the i = 1 case, where we recall the convention that $E_1(x) = E(x)$, and define

$$H(\mathbf{x}) := DE_i(\overline{\mathbf{x}})^{-1}D^{-1}, \quad \text{with} \quad D = \text{diag}((-1)^{i-1})_{i=1}^{N}$$

We can compute that $H(\mathbf{x}) = \sum_{1 \leqslant i \leqslant j \leqslant N} x_i x_{i+1} \cdots x_j E_{ij}$. Generally, for $k \geqslant 1$, and $\mathbf{x}=(x_k, \cdots, x_n)$,we define

$$H_k(\mathbf{x}) := \begin{pmatrix} I_{k-1} & 0 \\ 0 & H(\mathbf{x}) \end{pmatrix}$$

The equation (14) can be written as

$$H_i(\mathbf{x})H_i(\mathbf{a}) = H_{i+1}(\mathbf{b})H_i(\tilde{\mathbf{x}}) \tag{17}$$

Similar to $E(\mathbf{y}^1, \cdots, \mathbf{y}^k)$, we define the general form

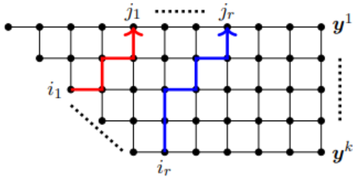
$$H(\mathbf{y}^1, \cdots, \mathbf{y}^k) := H_k(\mathbf{y}^k) \cdots H_1(\mathbf{y}^1), \text{where } \mathbf{y}^i = (y_i^i, y_{i+1}^i, \cdots, y_N^i).$$

Geometric RSK via a matrix formulation

Let rows $i_1 < \cdots < i_r$ and columns $j_1 < \cdots < j_r$. We can get the following equation:

$$\det H(\mathbf{y}^1, \cdots, \mathbf{y}^k)^{i_1,\cdots,i_r}_{j_1,\cdots,j_r} = \sum_{\pi_1 \cdots \pi_r} \mathbf{wt}(\pi_1) \cdots \mathbf{wt}(\pi_r) \tag{18}$$

where the sum is over up-right, non crossing paths, starting at locations $i_1, \cdots, i_r$ at the bottom border (including possibly the diagonal part) and ending at locations $j_1, \cdots, j_r$ at the top row of the grid.



Each vertex (a, b) of the grid is assigned a weight y^a_b and in this case the total weight of a path is $\mathbf{wt}(\pi) = \prod_{(a,b)\in\pi} y^a_b$

Geometric RSK via a matrix formulation

Theorem 3.3

Given a matrix $X := (x_j^i)_{1\leqslant j\leqslant N}^{1\leqslant i\leqslant n} := (\mathbf{x}^1, \cdots, \mathbf{x}^n)^{\mathbf{T}}$ the matrix equation

$$H(\mathbf{x}^1)H(\mathbf{x}^2)\cdots H(\mathbf{x}^n) = H_k(\mathbf{y}^k)H_{k-1}(\mathbf{y}^{k-1})\cdots H_1(\mathbf{y}^1), k = \min(n, N) \quad (19)$$

has a unique solution $(\mathbf{y}^1, \cdots, \mathbf{y}^k)$ with $\mathbf{y}^i := (y_i^i, \cdots, y_N^i)$, given by

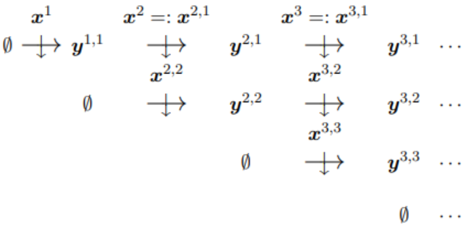
$$y_i^i = \frac{\tau_i^i}{\tau_i^{i-1}} \quad \text{and} \quad y_j^i = \frac{\tau_j^i \tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i} \quad \text{for} \quad i < j, \quad (20)$$

where

$$\tau_j^i := \sum_{\pi_1\cdots\pi_r \in \prod_{j-i+1,\cdots,j}^{1,\cdots,i}} \mathbf{wt}(\pi_1)\cdots\mathbf{wt}(\pi_r) \quad (21)$$

is a partition function corresponding to an ensemble of i non intersecting, down-right paths $\pi_1, \cdots, \pi_i$, along the entries of X, starting from $(1,1), \cdots, (1,i)$ and ending at $(n, j-i+1), \cdots, (n, j)$, respectively, with the weight of a path π_r given by $\mathbf{wt}(\pi_r) := \prod_{(a,b)\in\pi_r} x_b^a$

Let us now describe how **gRSK** can be encoded in this matrix formulation. We will make reference to the following diagram:



where $\mathbf{x}^i = (x_1^i, \cdots, x_N^i)$ for $i \leqslant 1$, is a sequence of words, which are successively row inserted via **gRSK**.

This procedure continues during the first N insertions at which stage the resulting tableau will have full depth of N rows.

After first N insertions, no additional rows will be created in the subsequent tableaux and the process continues as follows:

$$
\begin{array}{ccccccc}
 & \boldsymbol{x}^{N+1} =: \boldsymbol{x}^{N+1,1} & & \boldsymbol{x}^{N+2} =: \boldsymbol{x}^{N+2,1} & & \boldsymbol{x}^{N+3} =: \boldsymbol{x}^{N+3,1} & \\
\boldsymbol{y}^{N,1} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+1,1} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+2,1} & \rightarrow\!\!\!\!\!\downarrow & \cdots \\
 & \boldsymbol{x}^{N+1,2} & & \boldsymbol{x}^{N+2,2} & & \boldsymbol{x}^{N+3,2} & \\
\boldsymbol{y}^{N,2} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+1,2} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+2,2} & \rightarrow\!\!\!\!\!\downarrow & \cdots \\
 & \boldsymbol{x}^{N+1,3} & & \boldsymbol{x}^{N+2,3} & & \boldsymbol{x}^{N+3,3} & \\
\boldsymbol{y}^{N,3} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+1,3} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+2,3} & \rightarrow\!\!\!\!\!\downarrow & \cdots \\
 & \boldsymbol{x}^{N+1,4} & & \boldsymbol{x}^{N+2,4} & & \boldsymbol{x}^{N+3,4} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \boldsymbol{x}^{N+1,N} & & \boldsymbol{x}^{N+2,N} & & \boldsymbol{x}^{N+3,N} & \\
\boldsymbol{y}^{N,N} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+1,N} & \rightarrow\!\!\!\!\!\downarrow & \boldsymbol{y}^{N+2,N} & \rightarrow\!\!\!\!\!\downarrow & \cdots \\
 & \emptyset & & \emptyset & & \emptyset &
\end{array}
$$

Overall, we can get the following matrix equation:

$$\begin{cases} H(\mathbf{x}^1)H(\mathbf{x}^2)\cdots H(\mathbf{x}^n) = H_n(\mathbf{y}^{n,n})H_{n-1}(\mathbf{y}^{n,n-1})\cdots H_1(\mathbf{y}^{n,1}), n \leqslant N \\ H(\mathbf{x}^1)H(\mathbf{x}^2)\cdots H(\mathbf{x}^n) = H_N(\mathbf{y}^{n,n})H_{n-1}(\mathbf{y}^{n,n-1})\cdots H_1(\mathbf{y}^{n,1}), n \leqslant N \end{cases} \tag{22}$$

Geometric RSK via a matrix formulation

Theorem 3.4

Consider a matrix $X := (x_j^i : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant N)$ with nonnegative entries and denote by $(\mathbf{x}^1, \cdots, \mathbf{x}^n)$ its rows and by $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ its columns. Then there exists a one-to-one correspondence between $\mathbf{X}$ and set of variables $\mathbf{p}^i := (p_i^i, \cdots, p_N^i)$ for $i = 1, \cdots, \min(n, N)$ and $\mathbf{q}^i := (q_i^i, \cdots, q_n^i)$ for $i = 1, \cdots, \min(n, N)$, which are uniquely determined via equations.

$$H(\mathbf{x}^1)H(\mathbf{x}^2)\cdots H(\mathbf{x}^n) = H_k(\mathbf{p}^k)H_{k-1}(\mathbf{p}^{k-1})\cdots H_1(\mathbf{p}^1), k = \min(n, N) \tag{23}$$

$$H(\mathbf{x}_1)H(\mathbf{x}_2)\cdots H(\mathbf{x}_n) = H_k(\mathbf{q}^k)H_{k-1}(\mathbf{q}^{k-1})\cdots H_1(\mathbf{q}^1), k = \min(n, N) \tag{24}$$

Variables (p_j^i) and (q_j^i) are given in terms of the input variables (x_j^i) via relations (19) and (21).

For a full proof of this theorem, we refer to [4], Section 3 and Theorem 3.8.

Geometric RSK on Geometric Gelfand-Tsetlin patterns

Consider $\mathbf{Z} = (z^i_j)_{1 \leqslant j \leqslant i \leqslant n, j \leqslant n \wedge N}$, $\mathbf{Z'} = (z^i_j)'_{1 \leqslant j \leqslant i \leqslant n, j \leqslant n \wedge N}$. We set

$$z^i_j := p^j_j p^j_{j+1} \cdots p^j_{i-1} p^j_i \text{ for } 1 \leqslant j \leqslant i \leqslant n, j \leqslant n \wedge N$$
$$(z^i_j)' := q^j_j q^j_{j+1} \cdots q^j_{i-1} q^j_i \text{ for } 1 \leqslant j \leqslant i \leqslant n, j \leqslant n \wedge N$$

where $\mathbf{p}^1, \cdots, \mathbf{p}^k$ and $\mathbf{q}^1, \cdots, \mathbf{q}^k$ are as in Theorem 3.4.

Theorem 3.4 establishes a bijection between matrices $X = (x^i_j)_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant N}$ with nonnegative entries and a pair $(\mathbf{Z}, \mathbf{Z'}) = \mathbf{gRSK}(X)$.

we obtain that variables z^i_j are given in terms of ratios of partition functions from (20):

$$z^i_j = \frac{\tau^j_i}{\tau^{j-1}_i}$$
$$\tau^j_i := \sum_{\pi_1 \cdots \pi_r \in \prod^{1, \cdots, j}_{i-j+1, \cdots, i}} \mathbf{wt}(\pi_1) \cdots \mathbf{wt}(\pi_r)$$

Passage to standard combinatorial RSK setting

Replacing in (12) variables $\xi_k, \tilde{\xi}_k, a_k, b_k$ by $e^{\xi_k/\varepsilon}, e^{\tilde{\xi}_k/\varepsilon}, e^{a_k/\varepsilon}, e^{b_k/\varepsilon}$, taking the log on both sides of each relation therein and multiplying by ε, the set of equations (12) writes as

$$\begin{aligned}
&\tilde{\xi}_i = \xi_i + a_i \\
&\tilde{\xi}_k = a_k + \varepsilon \log(e^{\tilde{\xi}_{k-1}/\varepsilon} + e^{\xi_k/\varepsilon}), i+1 \leqslant k \leqslant N \\
&b_k = a_k + (\xi_k - \xi_{k-1}) - (\tilde{\xi}_k - \tilde{\xi}_{k-1}), \qquad i+1 \leqslant k \leqslant N
\end{aligned} \tag{25}$$

Taking now the limit $\varepsilon \to 0$ these reduce to the piecewise linear transformations (8) defining the standard **RSK** correspondence.
Replacing also the variables x^i_j, p^i_j, q^i_j in Theorem3.4 by $e^{x^i_j/\varepsilon}, e^{p^i_j/\varepsilon}, e^{q^i_j/\varepsilon}$ we obtain in the limit $\varepsilon \to 0$ the **RSK** correspondence, in the sense that variables (p^i_j) and (q^i_j) encode the **P** and **Q** tableaux of the standard **RSK**.

Passage to standard combinatorial RSK setting

In particular, the solution to the degeneration, as $\varepsilon \to 0$, of problem (23) is given via the degeneration of relations (20), (21) as:

$$p_i^i = \sigma_i^i - \sigma_i^{i-1} \text{ and } p_j^i = \sigma_j^i + \sigma_{j-1}^{i-1} - \sigma_j^{i-1} - \sigma_{j-1}^i \text{ for } i < j,$$

$$\text{with } \sigma_j^i := \max_{\pi_1 \cdots \pi_r \in \prod_{j-i+1,\cdots,i}^{1,\cdots,i}} \sum_{k=1}^{i} \mathbf{wt}(\pi_k)$$

being last passage percolation functionals corresponding to ensembles of i non intersecting, down-right paths $\pi_1, \cdots, \pi_i$, starting from $(1,1), \cdots, (1,i)$ and ending at $(n, j-i+1), \cdots, (n,j)$, respectively. The weight of a path π_r in this case is $\mathbf{wt}(\pi_r) := \sum_{(a,b)\in\pi_r} x_b^a$.

Passage to standard combinatorial RSK setting

Passing to the Gelfand-Tsetlin variables, we set

$$z_j^i := p_j^j + p_{j+1}^j + \cdots + p_{i-1}^j + p_i^j = \sigma_i^j - \sigma_{i-1}^j$$

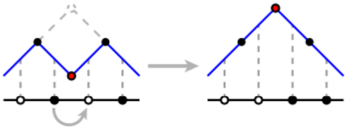
for $1 \leqslant j \leqslant i \leqslant n$,and $j \leqslant n \wedge N$. From this we get that

$$z_1^N + \cdots + z_j^N := \sigma_N^j$$

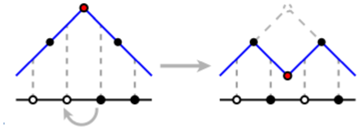
which in the case $j = 1$ is Schensted's theorem and for $j > 1$ is Greene's theorem.

Overview

- **Asymmetric simple exclusion process (ASEP)**:



(A)

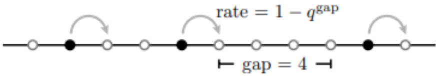


(B)

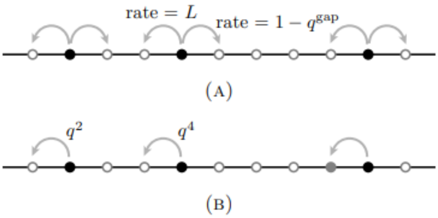
Particles in this model jump to the right with rate p and to the left with rate q such that $p + q = 1$, following the exclusion rule.

- **TASEP** (the totally asymmetric version of **ASEP**, i.e. $\mathbf{p = 1}$)

- **Q-TASEP** (Simple traffic model):



- **Q-PushASEP** :



The growth process

- **Definition**: We follow the standard practice of ordering particles from the right; for right-finite data the rightmost particle is labelled 1. Let us denote positions of particles at time $t > 0$ by
$$X_t(1) > X_t(2) > X_t(3) > \cdots,$$
where $X_t(i) \in \mathbb{Z}$ is the position of the i-th particle.
- The **TASEP** height function is a random walk path $h_t(z+1) = h_t(z) + \eta_t(z)$ with $\eta_t(z) = 1$ if there is a particle at z at time t and -1 if there is no particle at z at time t.
- We have
$$h_t(z) = -2(X_t^{-1}(z-1) - X_0^{-1}(-1)) - z,$$
which fixes $h_0(0) = 0.X_t^{-1}(u) = \min\{k \in \mathbb{Z} : X_t(k) \leqslant u\}$.

TASEP's Overview

- First, we prove **the Schütz's formula**, which gives the transition probability of TASEP particles in determinantal form. And in turn we can derive $\mathbb{P}_{\mathbf{y}}(X_t(k) \geqslant a)$.
  Remark: The Schütz's formula is not suitable for asymptotic analysis of TASEP, because the size of the matrix goes to ∞ as the number of particles N increases.
- Next, We provide some results on **Determinantal point process**. At the same time to prepare for finding correlation kernel.
- Finally, We use **Non-intersecting random walks** and find **correlation kernel** to get the **asymptotic analysis of TASEP**.

Distribution function of TASEP

Let $\mathbf{x} \in \Omega_N = \{x_1 > \cdots > x_N\} \subseteq \mathbb{Z}^N$, where Ω_N is called the Weyl chamber.

Proposition 4.1 (Schütz's formula)

The transition probability for $2 \leqslant N < \infty$ **TASEP** particles has a determinantal form:

$$\mathbb{P}(X_t = \mathbf{x} | X_0 = \mathbf{y}) = \det[F_{i-j}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leqslant i,j \leqslant N} \tag{26}$$

with $\mathbf{x}, \mathbf{y} \in \Omega_N$, and

$$F_n(x,t) = \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(1-w)^{-n}}{w^{x-n+1}} \mathrm{e}^{t(w-1)} \mathrm{d}w, \tag{27}$$

where $\Gamma_{0,1}$ is any simple loop oriented anticlockwise which includes $w = 0$ and $w = 1$.

Distribution function of TASEP

- **Approach**: **Bethe ansatz** (from quantum integrable system), an ansatz to diagonalize certain matrices.
- **Proof**: Four steps.
  · Consider $N > 2$ particles in **TASEP** and derive the master equation (**Kolmogorov forward equation**) for the process $X_t = (X_t(1), \cdots, X_t(N)) \in \Omega_N$.
  · Find a **general solution** to the Kolmogorov forward equation.
  · Find a **particular solution** which satisfies the **boundary** and **initial** conditions.

- **Notations and Operators**:
  - $P_t^{(N)}(\mathbf{y},\mathbf{x}) := \mathbb{P}(X_t = \mathbf{x} | X_0 = \mathbf{y})$.
  - $(\mathcal{L}^{(N)}F)(\mathbf{x}) = -\sum_{k=1}^{N} \mathbf{1}_{\{x_k - x_{k+1} > 1\}} (\nabla_k^- F)(\mathbf{x})$,

  where $F : \Omega_N \to \mathbb{R}, x_{N+1} = -\infty$.
  - ∇_k^- is the discrete derivative

$$\nabla^- f(z) = f(z) - f(z-1),\ f : \mathbb{Z} \to \mathbb{R}$$

  acting on the k-th argument of f.

- **Step1: Construct Kolmogorov forward equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t^{(N)}(\mathbf{y}, \cdot) = \mathcal{L}^{(N)} P_t^{(N)}(\mathbf{y}, \cdot), \; P_0^{(N)}(\mathbf{y}, \cdot) = \delta_{\mathbf{y}, \cdot}. \tag{28}$$

The idea is to rewrite (28) as a differential equation with constant coefficients and boundary conditions, **i.e.** if $u_t^{(N)} : \mathbb{Z}^N \to \mathbb{R}$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t^{(N)}(\mathbf{x}) = -\sum_{k=1}^{N} \nabla_k^- u_t^{(N)}(\mathbf{x}), \; u_0^{(N)}(\mathbf{x}) = \delta_{\mathbf{y},\mathbf{x}}, \tag{29}$$

with the boundary conditions

$$\nabla_k^- u_t^{(N)}(\mathbf{x}) = 0, \text{ when } x_k = x_{k+1} + 1 \tag{30}$$

then for $\mathbf{x}, \mathbf{y} \in \Omega_N$ one has with the boundary conditions

$$P_t^{(N)}(\mathbf{y}, \mathbf{x}) = u_t^{(N)}(\mathbf{x}) \tag{31}$$

- **Step2: Find a general solution to Kolmogorov forward equation**: Consider indistinguishable particles

$$\sum_{\sigma\in\mathbb{S}_N} u_t^{(N)}(\mathbf{x}_\sigma)\ ,$$

where $\mathbb{S}_N$ is the symmetric group and $\mathbf{x}_\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(N)})$. With this in mind we define the generating function

$$\phi_t^{(N)}(\mathbf{w}) = \frac{1}{|\mathbb{S}_N|} \sum_{\mathbf{x}\in\mathbb{Z}^N} \sum_{\sigma\in\mathbb{S}_N} \mathbf{w}^{\mathbf{x}_\sigma} u_t^{(N)}(\mathbf{x}_\sigma)\ ,$$

where $\mathbf{w}\in\mathbb{C}^N, \mathbf{w}^{\mathbf{x}} = z_1^{x_1}\cdots z_N^{x_N}$ and $|\mathbb{S}_N| = N!$. Since we would like the identity (31) to hold, it is reasonable to assume that $|u_t^{(N)}(\mathbf{x})| \leqslant \min_i \frac{t^{x_i-y_i}}{(x_i-y_i)!}$, which guarantees locally absolute convergence of the sum above and all the following computations. Now we can get the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{(N)}(\mathbf{w}) = -\phi_t^{(N)}(\mathbf{w})\sum_{k=1}^{N}(w_k-1) \tag{32}$$

(Continued from above)

- By ODE theory, We conclude that

$$\phi_t^{(N)}(\mathbf{w}) = C(\mathbf{w}) \prod_{k=1}^{N} \mathrm{e}^{(w_k-1)t} ,$$

for a function $C : \mathbb{C}^N \to \mathbb{C}$ which is independent of t, but can depend on $\mathbf{y}$. Then Cauchy's integral theorem gives a solution to Kolmogorov forward equation

$$\begin{aligned} u_t^{(N)}(\mathbf{x}) &= \frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} \frac{\phi_t^{(N)}(\mathbf{w})}{\mathbf{w}_\sigma^{\mathbf{x}+1}} \mathrm{d}\mathbf{w} \\ &= \frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} C(\mathbf{w}) \prod_{k=1}^{N} \frac{\mathrm{e}^{(w_k-1)t}}{w_{\sigma(k)}^{x_k+1}} \mathrm{d}\mathbf{w} \qquad (33) \end{aligned}$$

where $\mathbf{x}+1 = (x_1+1, \cdots, x_N+1)$ and Γ_0 is a contour in $\mathbb{C}^N$ around the origin.

- **Step3: Find a particular solution which satisfies the boundary conditions.**:
  We are going to find functions C and a contour Γ_0 such that the solution (33) satisfies the boundary conditions (30). More precisely, we consider functions $C_\sigma(z)$ to $C(z)$.
  In the case $x_k = x_{k+1} + 1$, the boundary condition (30) yields

$$\nabla_k^- u_t^{(N)}(\mathbf{x}) = -\frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} \prod_{i \neq k, k+1} \frac{C_\sigma(\mathbf{w})}{w_{\sigma(i)}^{x_i+1}} \frac{1 - w_{\sigma(k)}^{-1}}{w_{\sigma(k)}^{x_k} w_{\sigma(k+1)}^{x_k+1}} \prod_{j=1}^{N} \mathrm{e}^{(w_j-1)t} d\mathbf{w}$$

$$= -\frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} \prod_{i \neq k, k+1} \frac{C_\sigma(\mathbf{w})}{w_{\sigma(i)}^{x_i+1}} \frac{f(w_{\sigma(k)})}{(w_{\sigma(k)} w_{\sigma(k+1)})^{x_k}} \prod_{j=1}^{N} \mathrm{e}^{(w_j-1)t} d\mathbf{w}$$

$$= 0,$$

where $f(w_{\sigma(k)}) = 1 - w_{\sigma(k)}^{-1}$.

Distribution function of TASEP

(Continued from above)

- For all $\mathbf{w} \in \mathbb{C}^N$, we have

$$\sum_{\sigma \in \mathbb{S}_N} \frac{C_\sigma(\mathbf{w}) f(w_{\sigma(k)})}{(w_{\sigma(k)} w_{\sigma(k+1)})^{x_k}} = 0 \ ,$$

  Let $T_k \in \mathbb{S}_N$ be the transposition $(k, k+1)$, **i.e.** it interchanges the elements k and $k+1$. Then the above identity holds if we have

$$C_\sigma(\mathbf{w}) f(w_{\sigma(k)}) + C_{T_k \sigma}(\mathbf{w}) f(w_{\sigma(k+1)}) = 0. \qquad (34)$$

  Choose $C_\sigma(\mathbf{w})$ to be

$$C_\sigma(\mathbf{w}) = \operatorname{sgn}(\sigma) \prod_{i=1}^{N} f(w_{\sigma(i)})^i \phi(\mathbf{w}), \qquad (35)$$

  which satisfies the identity (34) for any function $\phi : \mathbb{C}^N \to \mathbb{R}$.

Distribution function of TASEP

- **Step4: Find a particular solution to satisfy the initial conditions**: Combining (33) with (29), the initial condition at $t = 0$ is given by

$$\frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} \frac{C_\sigma(\mathbf{w})}{\mathbf{w}_\sigma^{\mathbf{x}+1}} \mathrm{d}\mathbf{w} = \delta_{\mathbf{y},\mathbf{x}}. \tag{36}$$

If $\sigma = id \in \mathbb{S}_N$ and $C_{id}(\mathbf{w}) = \mathbf{w}^{\mathbf{y}}$, it will satisfy the identity (36). So choose the function ϕ in (35) to be

$$\phi(\mathbf{w}) = f(w_i)^{-i} w_i^{y_i}.$$

Thus, a candidate for the solution is given by

$$u_t^{(N)}(\mathbf{x}) = \frac{1}{(2\pi \mathrm{i})^N} \sum_{\sigma \in \mathbb{S}_N} \oint_{\Gamma_0} \mathrm{sgn}(\sigma) \prod_{k=1}^{N} \frac{f(w_k)^{k-\sigma(k)} \mathrm{e}^{(w_k-1)t}}{w_{\sigma(k)}^{x_k - y_{\sigma(k)}+1}} \mathrm{d}\mathbf{w}$$

which can be written as Schütz's formula (26). It is obvious that the contour Γ_0 should go around 0 and 1, since otherwise the determinant in (26) will vanish when **x** and **y** are far enough.

(Continued from above)

- Prove that the solution satisfies the initial condition. We have
$$F_n(x,0) = \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(1-w)^n}{w^{x+n+1}} \mathrm{d}w \,,$$
which in particular implies that $F_n(x,0) = 0$ for $x < -n$ and $x > 0$, and $F_0(x,0) = \delta_{x,0}$.

  After computing, we obtain that the matrix is upper-triangular with delta functions at the diagonal, which gives us the claim.

From Schütz's formula, let $\Gamma_{0,1} = B(0, R), R > 1$. we can derive $\mathbb{P}_{\mathbf{y}}(X_t(k) \geqslant a)$.

$$
\begin{aligned}
\mathbb{P}_{\mathbf{y}}(X_t(k) \geqslant a) &= \sum_{b=a}^{+\infty} \mathbb{P}(X_t = \mathbf{x} | X_0 = \mathbf{y}, X_t(k) = b) \\
&= \sum_{x_N=a}^{+\infty} \sum_{x_{N-1}=x_N+1}^{+\infty} \cdots \sum_{x_1=x_2+1}^{+\infty} \mathbb{P}(X_t = \mathbf{x} | X_0 = \mathbf{y}) \\
&= \sum_{x_N=a}^{+\infty} \sum_{x_{N-1}=x_N+1}^{+\infty} \cdots \sum_{x_2=x_3+1}^{+\infty} \det(F_{1i}, F_{2i}, \cdots, \sum_{x_1=x_2+1}^{+\infty} F_{Ni}) \\
&\vdots \\
&= \det(F_{1i}, F_{2i}, \cdots, \sum_{x_N=a}^{+\infty} \sum_{x_{N-1}=x_N+1}^{+\infty} \cdots \sum_{x_1=x_2+1}^{+\infty} F_{Ni})
\end{aligned} \tag{37}
$$

Distribution function of TASEP

(Continued from above)

From $\sum_{x_1=x_2+1}^{+\infty} w^{-x_1} = \frac{w^{-x_2}}{w-1}$, we have that

$$\sum_{x_1=x_2+1}^{+\infty} F_{1i} = \sum_{x_1=x_2+1}^{+\infty} \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(w-1)^{i-N}}{w^{x_1-y_{n+1-i}+i+1-N}} \mathrm{e}^{t(w-1)} \mathrm{d}w$$

$$= \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(w-1)^{i-N}}{w^{-y_{n+1-i}+i+1-N}} \mathrm{e}^{t(w-1)} \sum_{x_1=x_2+1}^{+\infty} w^{-x_1} \mathrm{d}w$$

$$= \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(w-1)^{i-N-1}}{w^{x_2-y_{n+1-i}+i+1-N}} \mathrm{e}^{t(w-1)} \mathrm{d}w$$

Repeating the last process, we have that

$$\sum_{x_N=a}^{+\infty} \sum_{x_{N-1}=x_N+1}^{+\infty} \cdots \sum_{x_1=x_2+1}^{+\infty} F_{Ni} = \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(w-1)^{i-2N}}{w^{a-1-y_{n+1-i}+i+1-N}} \mathrm{e}^{t(w-1)} \mathrm{d}w.$$

Combine (37), we derive that

$$\mathbb{P}_{\mathbf{y}}(X_t(k) \geqslant a) = \det\left(F_{1i}, F_{2i}, \cdots, \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{(w-1)^{i-2N}}{w^{a-1-y_{n+1-i}+i+1-N}} \mathrm{e}^{t(w-1)} \mathrm{d}w\right).$$

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Thank you !